ON SEPARABLY INJECTIVE BANACH SPACES

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ABSTRACT. In this paper we deal with two weaker forms of injectivity which turn out to have a rich structure behind: separable injectivity and universal separable injectivity. We show several structural and stability properties of these classes of Banach spaces. We provide natural examples of (universally) separably injective spaces, including $\mathcal{L}_\infty$ ultraproducts built over countably incomplete ultrafilters, in spite of the fact that these ultraproducts are never injective. We obtain two fundamental characterizations of universally separably injective spaces: a) A Banach space E is universally separably injective if and only if every separable subspace is contained in a copy of ℓ_∞ inside E. b) A Banach space E is universally separably injective if and only if for every separable space S one has $\mathrm{Ext}(\ell_\infty/S, E) = 0$. The final Section of the paper focuses on special properties of 1-separably injective spaces. Lindenstrauss obtained in the middle sixties a result that can be understood as a proof that, under the continuum hypothesis, 1-separably injective spaces are 1-universally separably injective; he left open the question in ZFC. We construct a consistent example of a Banach space of type $C(K)$ which is 1-separably injective but not 1-universally separably injective.

1. INTRODUCTION

A Banach space E is said to be injective if for every Banach space X and every subspace Y of X, each operator $t\colon Y \to E$ admits an extension $T\colon X \to E$. And E is said to be λ-injective if, besides, T can be chosen so that $\|T\| \leq \lambda\|t\|$. The space $\ell_i nfty$ is the best known example of 1-injective space. The two basic facts about injective spaces are that 1-injective spaces are isometric to $C(K)$-spaces with K extremely disconnected and that is not known if all injective spaces are isomorphic to 1-injective spaces.

In this paper we deal with two weaker forms of injectivity which turn out to have a rich structure behind: separable injectivity and universal separable injectivity. A Banach space E is said to be separably injective if it satisfies the extension property in the definition of injective spaces under the restriction that X is separable; and it is said to be universally separably injective if it satisfies the extension property when Y is separable. Obviously, injective spaces are universally separably injective and these, in turn, are separably injective; the converse implications fail.

The basic structural and stability properties of these classes are studied in Section 3: we will show that infinite-dimensional separably injective spaces are $\mathcal{L}_\infty$-spaces, contain c_0 and have Pełczyński's property (V). Universally separably injective spaces, moreover, are Grothendieck spaces, contain ℓ_∞ and enjoy Rosenthal's property (V). In Section 4 we provide natural examples of (universally) separably injective spaces, including the remarkable fact that ultraproducts built over countably incomplete ultrafilters are universally separably injective as long as they are $\mathcal{L}_\infty$-spaces, in spite of the fact that they are never injective. In Section 5 we obtain two fundamental characterizations of universally

separably injective spaces: a) A Banach space E is universally separably injective if and only if every separable subspace is contained in a copy of ℓ_∞ inside E. b) A Banach space E is universally separably injective if and only if for every separable space S one has $\mathrm{Ext}(\ell_\infty/S, E) = 0$; i.e., E is complemented in any superspace Z such that $Z/E = \ell_\infty/S$. Characterization (a) allows to prove that universal separable injectivity is a 3-space property, which provides many new examples of spaces with this property. Characterization (b)leads to the result $\mathrm{Ext}(\ell_\infty/c_0, \ell_\infty/c_0) = 0$, which provides a new unexpected solution for equation $\mathrm{Ext}(X, X) = 0$. The final Section 6 focuses on special properties of 1-separably injective spaces. This is the point in which set theory axioms enter the game. Indeed, Lindenstrauss obtained in the middle sixties what can be understood as a proof that, under the continuum hypothesis, 1-separably injective spaces are 1-universally separably injective; he left open the question in ZFC. We construct a consistent example of a Banach space of type $C(K)$ which is 1-separably injective but not 1-universally separably injective.

2. BACKGROUND

Our notation is fairly standard, as in [41]. Given a set Γ we denote by $\ell_\infty(\Gamma)$ the space of all bounded scalar functions on Γ, endowed with the sup norm and $c_0(\Gamma)$ is the closed subspace spanned by the characteristic functions of the singletons of Γ. A Banach space X is said to be an $\mathcal{L}_{p,\lambda}$-space (with $1 \le p \le \infty$ and $\lambda \ge 1$) if every finite dimensional subspace F of X is contained in another finite dimensional subspace of X whose Banach-Mazur distance to the corresponding ℓ_p^n is at most λ. A space X is said to be a $\mathcal{L}_p$-space if it is a $\mathcal{L}_{p,\lambda}$-space for some $\lambda \ge 1$; we will say that it is a $\mathcal{L}_{p,\lambda+}$-space when it is a $\mathcal{L}_{p,\lambda'}$-space for all $\lambda' > \lambda$. We are especially interested in $\mathcal{L}_\infty$ spaces. A Lindenstrauss space is one whose dual is isometric to $L_1(\mu)$ for some measure μ. Lindenstrauss spaces and $\mathcal{L}_{\infty,1+}$-spaces are identical classes. Throughout the paper, ZFC denotes the usual setting of set theory with the Axiom of Choice, while CH denotes the continuum hpothesis.

2.1. **The push-out and pull-back constructions.** The push-out construction appears naturally when one considers a couple of operators defined on the same space, in particular in any extension problem. Let us explain why. Given operators $\alpha : Y \to A$ and $\beta : Y \to B$, the associated push-out diagram is

$$
(1) \qquad
\begin{array}{ccc}
Y & \xrightarrow{\ \alpha\ } & A \\
{\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\beta'} \\
B & \xrightarrow{\ \alpha'\ } & \mathrm{PO}
\end{array}
$$

Here, the push-out space $\mathrm{PO} = \mathrm{PO}(\alpha, \beta)$ is quotient of the direct sum $A \oplus_1 B$ (with the sum norm) by the closure of the subspace $\Delta = \{(\alpha y, -\beta y) : y \in Y\}$. The map α' is given by the inclusion of B into $A \oplus_1 B$ followed by the natural quotient map $A \oplus_1 B \to (A \oplus_1 B)/\overline{\Delta}$, so that $\alpha'(b) = (0, b) + \overline{\Delta}$ and, analogously, $\beta'(a) = (a, 0) + \overline{\Delta}$.

The diagram (1) is commutative: $\beta'\alpha = \alpha'\beta$. Moreover, it is 'minimal' in the sense of having the following universal property: if $\beta'' : A \to C$ and $\alpha'' : B \to C$ are operators such that $\beta''\alpha = \alpha''\beta$, then there is a unique operator $\gamma : \mathrm{PO} \to C$ such that $\alpha'' = \gamma\alpha'$ and $\beta'' = \gamma\beta'$. Clearly, $\gamma((a, b) + \overline{\Delta}) = \beta''(a) + \alpha''(b)$ and one has $\|\gamma\| \le \max\{\|\alpha''\|, \|\beta''\|\}$.

Regarding the behaviour of the maps in Diagram 1, apart from the obvious fact that both α' and β' are contractive, we have:

Lemma 2.1.

(a) $\max \|\alpha'\|\, \|\beta'\| \le 1$.
(b) *If α is an isomorphic embedding, then Δ is closed.*
(c) *If α is an isometric embedding and $\|\beta\| \le 1$ then α' is an isometric embedding.*

(d) *If α is an isomorphic embedding then α' is an isomorphic embedding.*
(e) *If $\|\beta\| \leq 1$ and α is an isomorphism then α' is an isomorphism and*

$$\|(\alpha')^{-1}\| \leq \max\{1, \|\alpha\|\}.$$

Proof. (a) and (b) are clear. (c) If $\|\beta\| \leq 1$,

$$\|\alpha'(b)\| = \|(0,b) + \Delta\| = \inf_{y \in Y} \|\alpha y\| + \|b - \beta y\| \geq \inf_y \|\beta y\| + \|b - \beta y\| \geq \|b\|,$$

as required. (d) is clear after (c). (e) To prove the assertion about $(\alpha')^{-1}$, notice that for all $a \in A$ and $b \in B$ one has $(a, b) + \delta = (0, b + \beta y) + \delta$ for $y \in Y$ such that $\alpha y = a$. Therefore, for all $y' \in Y$ one has

$$
\begin{aligned}
\|b + \beta y\| &\leq \|b + \beta y + \beta y'\| + \|\beta y'\| \\
&\leq |b + \beta y + \beta y'\| + \|y'\| \\
&\leq \|b + \beta y + \beta y'\| + \|\alpha^{-1}\|\|\alpha y'\|
\end{aligned}
$$

from where the assertion follows. □

The pull-back construction is the dual of that of push-out in the sense of categories, that is, "reversing arrows". Indeed, let $\alpha : A \to Z$ and $\beta : B \to Z$ be operators acting between Banach spaces. The associated pull-back diagram is

(2)
$$
\begin{array}{ccc}
\text{PB} & \xrightarrow{\ '\beta\ } & A \\
{\scriptstyle '\alpha}\downarrow & & \downarrow{\scriptstyle \alpha} \\
B & \xrightarrow{\ \beta\ } & Z
\end{array}
$$

Here, the pull-back space is $\text{PB} = \text{PB}(\alpha, \beta) = \{(a, b) \in A \times B : \alpha(a) = \beta(b)\}$, where $A \times B$ carries the sup norm. The arrows after primes are the restriction of the projections onto the corresponding factor. Needless to say (2) is minimally commutative in the sense that if the operators $''\beta : C \to A$ and $''\alpha : C \to B$ satisfy $\alpha \circ ''\beta = \beta \circ ''\alpha$, then there is a unique operator $\gamma : C \to \text{PB}$ such that $''\beta = '\beta\gamma$ and $''\beta = '\beta\gamma$. Clearly, $\gamma(c) = (''\beta(c), ''\alpha(c))$ and $\|\gamma\| \leq \max\{\|''\alpha\|, \|''\beta\|\}$.

Quite clearly $'\alpha$ is onto if α is.

2.2. **Exact sequences.** A short exact sequence of Banach spaces is a diagram

(3)
$$0 \longrightarrow Y \xrightarrow{\ \imath\ } X \xrightarrow{\ \pi\ } Z \longrightarrow 0$$

where Y, X and Z are Banach spaces and the arrows are operators in such a way that the kernel of each arrow coincides with the image of the preceding one. By the open mapping theorem $\imath$ embeds Y as a closed subspace of X and Z is isomorphic to the quotient $X/\imath(Y)$.

We say that $0 \to Y \to X_1 \to Z \to 0$ is equivalent to (3) if there exists an operator $t : X \to X_1$ making commutative the diagram

(4)
$$
\begin{array}{ccccccccc}
0 & \longrightarrow & Y & \longrightarrow & X & \longrightarrow & Z & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle t} & & \| & & \\
0 & \longrightarrow & Y & \longrightarrow & X_1 & \longrightarrow & Z & \longrightarrow & 0.
\end{array}
$$

This is a true equivalence relation in view of the classical 'three-lemma' asserting that in a commutative diagram of vector spaces and linear maps

$$0 \longrightarrow Y \longrightarrow X \longrightarrow Z \longrightarrow 0$$
$$\quad u\downarrow \qquad t\downarrow \qquad v\downarrow$$
$$0 \longrightarrow Y_1 \longrightarrow X_1 \longrightarrow Z_1 \longrightarrow 0.$$

having exact rows, if u and v are surjective (resp., injective) then so is t. Hence, the operator t in Diagram 4 is a bijection and so it is a linear homeomorphism, according to the open mapping theorem.

The sequence (3) is said to be trivial if it is equivalent to the direct sum sequence $0 \to Y \to Y \oplus Z \to Z \to 0$. This happens if and only if it splits, that is, there is an operator $p : X \to Y$ such that $p\imath = \mathbf{1}_Y$ ($\imath(Y)$ is complemented in X); equivalently, there is an operator $s : Z \to X$ such that $\pi s = \mathbf{1}_Z$.

For every pair of Banach spaces Z and Y, we denote by $\mathrm{Ext}(Z, Y)$ the space of all exact sequences $0 \to Y \to X \to Z \to 0$ modulo equivalence. We write $\mathrm{Ext}(Z, Y) = 0$ to indicate that every sequence of the form (3) is trivial. The reason for this notation is that $\mathrm{Ext}(Z, Y)$ has a natural linear structure [14, 18] for which the (class of the) trivial extension is the zero element.

Suppose we are given an extension (3) and an operator $t : Y \to B$. Consider the push-out of the couple $(\imath, t)$ and draw the corresponding arrows:

$$0 \longrightarrow Y \xrightarrow{\imath} X \longrightarrow Z \longrightarrow 0$$
$$\quad t\downarrow \qquad t'\downarrow$$
$$\qquad B \xrightarrow{\imath'} \mathrm{PO}$$

By Lemma 2.1(a), $\imath'$ is an isomorphic embedding. Now, the operator $\pi : X \to Z$ and the null operator $n : B \to Z$ satisfy the identity $\pi\imath = nt = 0$, and the universal property of the push-out gives a unique operator $\varpi : \mathrm{PO} \to Z$ making the following diagram commutative:

(5)
$$0 \longrightarrow Y \xrightarrow{\imath} X \xrightarrow{\pi} Z \longrightarrow 0$$
$$\quad t\downarrow \qquad t'\downarrow \qquad \|$$
$$0 \longrightarrow B \xrightarrow{\imath'} \mathrm{PO} \xrightarrow{\varpi} Z \longrightarrow 0$$

Or else, just take $\varpi((x, b)+\Delta) = \pi(x)$, check commutativity, and discard everything but the definition of PO.

Elementary considerations show that the lower sequence in the preceding Diagram is exact. That sequence will we referred to as the push-out sequence. Actually, the universal property of the push-out makes this diagram unique, in the sense that for any other commutative diagram of exact sequences

$$0 \longrightarrow Y \xrightarrow{\imath} X \xrightarrow{\pi} Z \longrightarrow 0$$
$$\quad t\downarrow \qquad \downarrow \qquad \|$$
$$0 \longrightarrow B \longrightarrow X' \longrightarrow Z \longrightarrow 0$$

the lower row turns out to be equivalent to the push-out sequence in (5). For this reason we usually refer to a diagram like that as a push-out diagram. The universal property of the push-out diagram immediately yields

Lemma 2.2. *With the above notations, the push-out sequence splits if and only if t extends to X, that is, there is an operator $T : X \to B$ such that $T\imath = t$.*

Proceeding dually one obtains the pull-back sequence. Consider again (3) and an operator $u : A \to Z$. Let us form the pull-back diagram of the couple (π, u) thus:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & Y & \xrightarrow{\imath} & X & \xrightarrow{\pi} & Z & \longrightarrow & 0 \\
& & & & {}'u \uparrow & & \uparrow u & & \\
& & & & \mathrm{PB} & \xrightarrow{{}'\pi} & A & &
\end{array}
$$

Recalling that ${}'\pi$ is onto and taking $\jmath(y) = (0, \imath(y))$, it is easily seen that the following diagram is commutative:

(6)
$$
\begin{array}{ccccccccc}
0 & \longrightarrow & Y & \xrightarrow{\imath} & X & \xrightarrow{\pi} & Z & \longrightarrow & 0 \\
& & \| & & {}'u \uparrow & & \uparrow u & & \\
0 & \longrightarrow & Y & \xrightarrow{\jmath} & \mathrm{PB} & \xrightarrow{{}'\pi} & A & \longrightarrow & 0
\end{array}
$$

The lower sequence is exact, and we shall referred to it as the pull-back sequence. The splitting criterion is now as follows.

Lemma 2.3. *With the above notations, the pull-back sequence splits if and only if u lifts to X, that is, there is an operator $L : A \to X$ such that $\pi L = u$.*

Given an exact sequence $0 \longrightarrow Y \xrightarrow{\imath} X \xrightarrow{\pi} Z \longrightarrow 0$ and another Banach space B, taking operators with values in B one gets the exact sequence

$$
0 \longrightarrow \mathfrak{L}(Z, B) \xrightarrow{\pi^*} \mathfrak{L}(X, B) \xrightarrow{\imath^*} \mathfrak{L}(Y, B)
$$

(in which $\varkappa^*$ means composition with $\varkappa$ on the right) that can be continued to form a "long exact sequence"

$$
0 \longrightarrow \mathfrak{L}(Z, B) \xrightarrow{\pi^*} \mathfrak{L}(X, B) \xrightarrow{\imath^*} \mathfrak{L}(Y, B) \xrightarrow{\beta} \mathrm{Ext}(Z, B) \longrightarrow \mathrm{Ext}(X, B) \longrightarrow \mathrm{Ext}(Y, B)
$$

A detailed description of homology sequences can be seen in [14]. Here we just indicate the action of β: it takes $t \in \mathfrak{L}(Y, B)$ into (the class in $\mathrm{Ext}(Z, B)$ of) the lower extension of the push-out diagram (5).

3. BASIC PROPERTIES OF (UNIVERSALLY) SEPARABLY INJECTIVE SPACES

Definition 3.1. A Banach space E is separably injective if for every separable Banach space X and each subspace $Y \subset X$, every operator $t : Y \to E$ extends to an operator $T : X \to E$. If some extension T exists with $\|T\| \le \lambda \|t\|$ we say that E is λ-separably injective.

It is easy to check that every separably injective space E is λ-separably injective for some λ since every sequence of norm-one operators $t_n : Y_n \to E$ induces a norm-one operator $t : \ell_1(Y_n) \to E$. Separable injective spaces can be characterized as follows.

Proposition 3.2. *For a Banach space E the following properties are equivalent.*

(a) *E is separably injective.*
(b) *Every operator from a subspace of ℓ_1 into E extends to ℓ_1.*
(c) *For every Banach space X and each subspace Y such that X/Y is separable, every operator $t : Y \to E$ extends to X.*
(d) *If X is a Banach space containing E and X/E is separable, then E is complemented in X.*
(e) *For every separable space S one has $\mathrm{Ext}(S, E) = 0$.*

Moreover,

(1) *The space E is λ-complemented in every Z such that Z/E is separable if and only if every operator $t : Y \to E$ admits an extension $T : X \to E$ with $\|T\| \le \lambda\|t\|$, whenever X/Y is separable.*

(2) *If E is λ-separably injective, then for every operator $t : Y \to E$ there exists an extension $T : X \to E$ of T with $\|T\| \le 3\lambda\|t\|$, whenever X/Y is separable.*

Proof. It is clear that $(c) \Rightarrow (a) \Rightarrow (b)$ and $(c) \Rightarrow (d) \Leftrightarrow (e)$. Moreover, (1) shows that $(d) \Rightarrow (c)$ and (2) shows that $(a) \Rightarrow (c)$. The remaining implication $(b) \Rightarrow (a)$ follows from the proof of (2) below.

For the sufficiency statement in (1) simply consider t as the identity on E. For the necessity statement, given an operator $t : Y \to E$ form the associated push-out diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & Y & \xrightarrow{\imath} & X & \xrightarrow{\pi} & X/Y & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle t'} & & \| & & \\
0 & \longrightarrow & E & \xrightarrow{\imath'} & \mathrm{PO} & \longrightarrow & \mathrm{PO}/E & \longrightarrow & 0.
\end{array}
$$

Since $\mathrm{PO}/E = X/Y$ is separable, there is a projection $p : \mathrm{PO} \to E$ with norm at most λ, and thus, recalling that $\|t'\| \le 1$, the composition $pt' : X \to E$ yields an extension of t with norm at most λ.

The proof for (2) is a little more tricky. Let q be a surjective map from $\ell_1 \to X/Y$. The lifting property of ℓ_1 provides an operator $Q : \ell_1 \to X$. Consider thus the commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \ker q & \xrightarrow{\jmath} & \ell_1 & \xrightarrow{q} & X/Y & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle Q} & & \| & & \\
0 & \longrightarrow & Y & \longrightarrow & X & \longrightarrow & X/Y & \longrightarrow & 0
\end{array}
$$

Let us construct the true push-out of the couple $(\phi, \jmath)$ and the corresponding complete diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \ker q & \xrightarrow{\jmath} & \ell_1 & \xrightarrow{q} & X/Y & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi'} & & \| & & \\
0 & \longrightarrow & Y & \xrightarrow{\jmath'} & \mathrm{PO} & \longrightarrow & X/Y & \longrightarrow & 0.
\end{array}
$$

We can consider without loss of generality that $\|\phi\| = 1$. Let $S : \ell_1 \to E$ be an extension of $t\phi$ with $\|S\| \le \lambda\|t\phi\| \le \lambda\|t\|$. By the universal property of the push-out, there exists an operator $L : \mathrm{PO} \to E$ such that $L\phi' = S$ and $\|L\| \le \max\{\|t\|, \|S\|\} \le \lambda\|t\|$. Again by the universal property of the push-out, there is a diagram of equivalent exact sequences

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & Y & \xrightarrow{\jmath'} & \mathrm{PO} & \longrightarrow & X/Y & \longrightarrow & 0 \\
& & \| & & \downarrow{\scriptstyle \gamma} & & \| & & \\
0 & \longrightarrow & Y & \xrightarrow{\imath} & X & \xrightarrow{p} & X/Y & \longrightarrow & 0,
\end{array}
$$

where the isomorphism γ is defined as $\gamma((y, u) + \Delta) = \jmath(y) + Q(u)$ is such that $\|\gamma\| \le \max\{\|\jmath\|, \|Q\|\} \le 1$. The desired extension of t to X is $T = L\gamma^{-1}$, where γ^{-1} comes defined by

$$
\gamma^{-1}(x) = (x - s(px), s(px)) + \Delta,
$$

where $s : X/Y \to \ell_1$ is a homogeneous bounded selection for q with $\|s\| \le 1$. One clearly has $\|\gamma^{-1}\| \le 3$, and therefore $\|T\| \le 3\lambda$. $\qquad\square$

We are especially interested in the following subclass of separably injective spaces.

Definition 3.3. A Banach space E is said to be universally separably injective if for every Banach space X and each separable subspace $Y \subset X$, every operator $t : Y \to E$ extends to an operator $T : Y \to X$. If some extension T exists with $\|T\| \leq \lambda\|t\|$ we say that E is universally λ-separably injective.

It is easy to check that a Banach space E is universally separably injective if and only if every E-valued operator with separable range extends to any superspace. It is also easy to show that every universally separably injective space is λ-universally separably injective for some λ.

Recall that a Banach space X has Pełczyński's property (V) if each operator defined on X is either weakly compact or it is an isomorphism on a subspace isomorphic to c_0. We will say that X has Rosenthal's property (V) if it satisfies the preceding condition with ℓ_∞ replacing c_0. It is well-known that Lindenstrauss spaces (i.e., $\mathcal{L}_{\infty,1+}$-spaces) have this property [36].

Not all $\mathcal{L}_\infty$-spaces have Pełczyński's property (V): for example, the $\mathcal{L}_\infty$-spaces without copies of c_0 constructed by Bourgain and Delbaen [9]; or those that can be obtained from Bourgain-Pisier [11]; or the space Ω constructed in [15] as a twisted sum

$$0 \longrightarrow C[0,1] \longrightarrow \Omega \longrightarrow c_0 \longrightarrow 0$$

with strictly singular quotient map. Recall that a Banach space X is said to be a Grothendieck space if every operator from X to a separable Banach space (or to c_0) is weakly compact. Clearly, a Banach space with property (V) is a Grothendieck space if and only if it has no complemented subspace isomorphic to c_0. It is well-known that ℓ_∞ is a Grothendieck space. One moreover has.

Proposition 3.4.

(a) *A separably injective space is of type $\mathcal{L}_\infty$, has Pełczyński's property (V) and, when it is infinite dimensional, contains copies of c_0.*

(b) *A universally separably injective space is a Grothendieck space of type $\mathcal{L}_\infty$, has Rosenthal's property (V) and, when it is infinite dimensional, contains ℓ_∞.*

Proof. (a) Let E be a λ-separably injective space. We want to see that if Y is a subspace of any Banach space X, every operator $t : Y \to E$ extends to an operator $T : X \to E^{**}$ with $\|T\| \leq \lambda\|t\|$. This implies that E^{**} is λ-injective, by an old result of Lindenstrauss [39, Theorem 2.1]. Being of infinite dimension, E^{**} is an $\mathcal{L}_{\infty,9\lambda+}$ space and so is E. Let $t : Y \to E$ be an operator. Given a finite-dimensional subspace F of X, let $T_F : F \to E$ be any operator extending the restriction of t to $Y \cap F$. Let $\mathcal{F}$ be the set of finite-dimensional subspaces of X, ordered by inclusion, let $\mathcal{U}$ be any ultrafilter refining the Fréchet filter on $\mathcal{F}$, that is, containing every set of the form $\{G \in \mathcal{F} : F \subset G\}$ for fixed $F \in \mathcal{F}$. Then, define $T : X \to E^{**}$ taking

$$T(x) = \text{weak*-}\lim_{\mathcal{U}(F)} T_F(1_{F(x)}x).$$

It is easily seen that T is a linear extension of t, with $\|T\| \leq \lambda\|t\|$.

To show that E contains c_0 and has property (V), let $T : E \to X$ be a non-weakly compact operator (E being an infinite dimensional $\mathcal{L}_\infty$ space cannot be reflexive). Choose a bounded sequence (x_n) in E such that (Tx_n) has no weakly convergent subsequences and let Y be the subspace spanned by (x_n) in E. As Y is separable we can regard it as a subspace of $C[0,1]$. Let $J : C[0,1] \to E$ be any operator extending the inclusion of Y into E. Since $TJ : C[0,1] \to E$ is not weakly compact, TJ is an isomorphism on some subspace isomorphic to c_0; and the same occurs to T.

(b) If, in addition to that, E is universally separably injective we may take $T : E \to Z$ and $Y \subset E$ as before but this time we consider Y as a subspace of ℓ_∞. If $J : \ell_\infty \to E$ is any extension of the inclusion of Y into E, then $TJ : \ell_\infty \to Z$ is not weakly compact. Hence it is an isomorphism on some subspace isomorphic to ℓ_∞ and so is T. $\square$

Several modifications on the proof of Ostrovskii [44] yield

Proposition 3.5. *A λ-separably injective space with $\lambda < 2$ is either finite-dimensional or has density character at least $\mathfrak{c}$.*

Recall that a class of Banach spaces is said to have the 3-space property if whenever X/Y and Y belong to the class, then so X does. See the monograph [18].

Proposition 3.6.

(1) *The class of separably injective spaces has the 3-space property.*
(2) *The quotient of two separably injective spaces is separably injective.*
(3) *The class of universally separably injective spaces has the 3-space property.*
(4) *The quotient of a universally separably injective space by a separably injective space is universally separably injective.*

Proof. The simplest proof for the 3-space property (1) follows from characterization (2) in Proposition 3.2: let us consider an exact sequence $0 \longrightarrow F \longrightarrow E \overset{\pi}{\longrightarrow} G \longrightarrow 0$ in which both F and G are separably injective. Let $\phi : K \to E$ be an operator from a subspace $\imath : K \to \ell_1$ of ℓ_1; then $\pi\phi$ can be extended to an operator $\Phi : \ell_1 \to G$, which can in turn be lifted to an operator $\Psi : \ell_1 \to E$. The difference $\phi - \Psi\imath$ takes values in F and can thus be extended to an operator $e : \ell_1 \to F$. The desired operator is $\Psi + e$. A different homological proof that properties having the form $\text{Ext}(X, -) = 0$ are always 3-space properties can be found in [14].

To prove (2) and (4) let us consider an exact sequence $0 \longrightarrow F \longrightarrow E \overset{\pi}{\longrightarrow} G \longrightarrow 0$ in which F is separably injective and E is (universally) separably injective. Let $\phi : Y \to G$ be an operator from a separable space Y which is a subspace of a separable (arbitrary) space X. Consider the pull-back diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & F & \longrightarrow & E & \overset{q}{\longrightarrow} & G & \longrightarrow & 0 \\
& & \| & & \uparrow{\scriptstyle\Phi} & & \uparrow{\scriptstyle\phi} & & \\
0 & \longrightarrow & F & \longrightarrow & \text{PB} & \overset{Q}{\longrightarrow} & Y & \longrightarrow & 0
\end{array}
$$

Since F is separably injective, the lower exact sequence splits, so Q has a selection operator $s : Y \to$ PB. By the injectivity assumption about E, there exists an operator $T : X \to E$ agreeing with Qs on Y. Then $qT : X \to G$ is the desired extension of ϕ.

The proof for (3) has to wait until Theorem 5.1 when a suitable characterization of universally separably injective spaces will be presented. $\qquad\qquad\square$

Several variations of these results can be seen in [20]. It is obvious that if $(E_i)_{i\in I}$ is a family of λ-separably injective Banach spaces, then $\ell_\infty(I, E_i)$ is λ-separably injective. The non-obvious fact that also $c_0(I, E_i)$ is separably injective can be considered as a vector valued version of Sobczyk's theorem. Proofs for this result have been obtained by Johnson-Oikhberg [35], Rosenthal [48], Cabello [12] and Castillo-Moreno [19], each with its own estimate for the constant. These are $2\lambda^2$ (implicitly), $\lambda(1+\lambda)^+$, $(3\lambda^2)^+$ and $6\lambda^+$, respectively.

Remarks 3.7. Let $0 \to F \to E \to G \to 0$ be a short exact sequence of Banach spaces. We know from Proposition 3.6 that E is separably injective if the other two relevant spaces are; and the same happens with G. What about F? Bourgain [8] constructed an uncomplemented copy of ℓ_1 in ℓ_1, which yields an exact sequence $0 \to \ell_1 \to \ell_1 \to B \to 0$ that does not split. By Lindenstrauss' lifting B is not an $\mathcal{L}_1$ space. Its dual sequence $0 \to B^* \to \ell_\infty \to \ell_\infty \to 0$ shows that the kernel of a quotient mapping between two injective spaces may fail to be even an $\mathcal{L}_\infty$-space.

4. Examples

All injective spaces are universally separably injective. Sobczyk theorem states that c_0 –and $c_0(\Gamma)$, in general– are 2-separably injective in its natural supremum norm. They are not universally separably injective since they do not contain ℓ_∞.

4.1. Twisted sums.
The 3-space property yields that twisted sums of separably injective are also separably injective. In particular:

- Twisted sums of c_0 and $c_0(\Gamma)$. This includes the Johnson-Lindenstrauss spaces $C(\Delta_{\mathcal{M}})$ [34] obtained taking the closure of the linear span in ℓ_∞ of the characteristic functions $\{1_n\}_{n\in\mathbb{N}}$ and $\{1_{M_\alpha}\}_{\alpha\in J}$ for an uncountable almost disjoint family $\{M_\alpha\}_{\alpha\in J}$ of subsets of $\mathbb{N}$. Marciszewski and Pol answer in [43] a question of Koszmider [37, Question 5] showing that there exist $2^{\mathfrak{c}}$ almost disjoint families $\mathcal{M}$ generating non-isomorphic $C(\Delta_{\mathcal{M}})$-spaces.
- Twisted sums of two nonseparable $c_0(\Gamma)$ spaces. This includes variations of the previous construction using the Sierpinski-Tarski [38] generalization of the construction of almost-disjoint families; the Ciesielski-Pol space (see [24]); the WCG nontrivial twisted sums of $c_0(\Gamma)$ obtained independently by Argyros, Castillo, Granero, Jimenez and Moreno [6] and by Marciszewski [42]
- Twisted sums of c_0 and ℓ_∞, as those constructed in [13].
- A twisted sum of c_0 and $c_0(\ell_\infty/c_0)$ that is not complemented in any $C(K)$-space, as the one obtained in [20].

It is not hard to prove that none of the preceding examples can be universally separably injective.

4.2. The space $\ell_\infty^c(\Gamma)$.
A typical 1-universally separably injective space is the space $\ell_\infty^c(\Gamma)$ of countably supported bounded functions $f : \Gamma \to \mathbb{R}$, where Γ is an uncountable set. This space is isomorphic but not isometric to some $C(K)$ space, showing in this way that the theory of universally separably injective spaces does not run parallel with that of injective spaces. What makes this space universally separably injective space is the following property:

Definition 4.1. We say that a Banach space X is ℓ_∞-upper-saturated if every separable subspace of X is contained in some (isomorphic) copy of ℓ_∞ inside X.

It is clear that an ℓ_∞-upper-saturated space is universally separably injective. We will prove in Theorem 5.1 that the converse also holds.

4.3. The space ℓ_∞/c_0.
Since ℓ_∞ is injective and c_0 is separably injective, it follows from Proposition 3.6 that ℓ_∞/c_0 is universally separably injective, although the constant is not optimal. It follows from Proposition 4.6(a) that ℓ_∞/c_0 is 1-universally separably injective, hence –by Theorem 5.1– it is ℓ_∞-upper-saturated. This can be improved to show that every separable subspace of ℓ_∞/c_0 is contained in a subalgebra of ℓ_∞/c_0 isometrically isomorphic to ℓ_∞.

It is well-known that ℓ_∞/c_0 is not injective. The simplest proof appears in Rosenthal [47]: an injective space containing $c_0(I)$ must also contain $\ell_\infty(I)$; it is well-known that ℓ_∞/c_0 contains $c_0(I)$ for $|I| = \mathfrak{c}$ while it cannot contain $\ell_\infty(I)$. The proof is is quite rough in a sense: it says that ℓ_∞/c_0 is uncomplemented in its bidual, a huge superspace. Denoting $\mathbb{N}^* = \beta\mathbb{N} \setminus \mathbb{N}$, Amir had shown in [2] that $C(\mathbb{N}^*)$ is not complemented in $\ell_\infty(2^{\mathfrak{c}})$, which provides another proof that l_∞/c_0 is not injective. Amir's proof can be refined in order to get $C(\mathbb{N}^*)$ uncomplemented in a much smaller space. It can be shown [21] that $C(\mathbb{N}^*)$ contains an uncomplemented copy Y of itself. Corollary 5.5 yields that the corresponding quotient $C(\mathbb{N}^*)/Y$ cannot be isomorphic to a quotient of ℓ_∞ by a separable subspace.

4.4. **Other** $C(K)$**-spaces.** Recall that a compact Hausdorff space K is said to be an F-space if disjoint open F_σ sets (equivalently, cozeroes) have disjoint closures. Equivalently, if any continuous function $f : K \to \mathbb{R}$ can be decomposed as $f = u|f|$ for some continuous function $u : K \to \mathbb{R}$. One has (see [**?**] for a proof of this and several generalized forms of this result).

Proposition 4.2. *A $C(K)$ space is 1-separably injective if and only if K is an F-space.*

Simple examples show that when a $C(K)$-space is only isomorphic to a 1-separably injective then K does not need to be an F-space. It is an immediate consequence of Tietze's extension theorem that a closed subset of an F-space is an F-space. In particular, $\mathbb{N}^* = \beta\mathbb{N} \setminus \mathbb{N}$ is an F-space.

Given a compact space K, we write K' for its derived set, that is, the set of its accumulation points. This process can be iterated to define $K^{(n+1)}$ as $(K^{(n)})'$ with $K^{(0)} = K$. We say that K has height n if $K^{(n)} = \varnothing$. We say that K has finite height if it has height n for some $n \in \mathbb{N}$.

Proposition 4.3. *If K is a compact space of height n, then $C(K)$ is $(2n-1)$-separably injective. Consequently, if K is a compact space of finite height then $C(K)$ is separably injective although it is not universally separably injective.*

Proof. Let $Y \subset X$ with X separable and let $t : Y \to C(K)$ be a norm one operator. The range of t is separable and every separable subspace of a $C(K)$ is contained in an isometric copy of $C(L)$, where L is the quotient of K after identifying k and k' when $y(k) = y(k')$ for all $y \in Y$. This L is metrizable because Y is separable. Moreover, if K has height n, then L has height at most n and so it is homeomorphic to $[0, \omega^r \cdot k]$ with $r < n$, $k < \omega$. Since $C[0, \omega^r \cdot k]$ is $(2r+1)$-separably injective [7], our operator can be extended to an operator $T : X \to C(K)$ with norm

$$\|T\| \leq (2r+1)\|t\| \leq (2n-1)\|t\|,$$

concluding the proof. $\qquad\square$

When K is a metrizable compact of finite height n, Baker [7] showed that $2n-1$ is the best constant for separable injectivity, using arguments from Amir [2]. There are some difficulties in generalizing those arguments for nonmetrizable compact spaces, so we do not know if it could exist a nonmetrizable compact space K of height n such that $C(K)$ is λ-separably injective for some $\lambda < 2n-1$.

Proposition 4.4. *The space of all bounded Borel (respectively, Lebesgue) measurable functions on the line is 1-separably injective in the sup norm.*

Proof. Clearly the given spaces are in fact Banach algebras satisfying the inequality required by Albiac-Kalton characterization (see [1]). Thus they can be represented as $C(K)$ spaces. On the other hand, each measurable function can be decomposed as $f = u|f|$, with u (and $|f|$, of course) measurable. This clearly implies that the corresponding compacta are F-spaces. $\qquad\square$

Argyros proved in [5] that none of the spaces in the above example is injective. This is very simple in the Borel case: the characteristic functions of the singletons generate a copy of $c_0(\mathbb{R})$ in the space of bounded Borel functions. The density character of the latter space is the continuum, as there are $\mathfrak{c}$ Borel subsets. Therefore it cannot contain a copy of $\ell_\infty(\mathbb{R})$, whose density character is $2^{\mathfrak{c}}$.

4.5. **M-ideals.** A closed subspace $J \subset X$ is called an M-ideal [27, Definition 1.1] if its annihilator $J^\perp = \{x^* \in X^* : \langle x^*, x \rangle = 0 \quad \forall x \in J\}$ is an L-summand in X^*. This just means that there is a linear projection P on X^* whose range is $J^\perp$ and such that $\|x^*\| = \|P(x^*)\| + \|x^* - P(x^*)\|$ for all $x^* \in X^*$. The easier examples of M-ideals are just ideals in $C(K)$-spaces. In particular, if M is a closed subset of the compact space K and $L = K \setminus M$ one has that $C_0(L)$ is an M-ideal in $C(K)$ is straightforward from the Riesz representation of $C(K)^*$. A remarkable generalization of Borsuk-Dugundji theorem

for M-ideals was provided by Ando [4] and, independently, Choi and Effros [22]. In order to state it let us recall that a Banach space Z has the λ-approximation property (λ-AP, for short) if, for every $\varepsilon > 0$ and every compact subset K of Z, there exists a finite rank operator T on Z, with $\|T\| \leq \lambda$, such that $\|Tz - z\| < \varepsilon$, for every $z \in K$. We say that Z has the bounded approximation property (BAP for short) if it has the λ-AP, for some λ.

We refer the reader to [17] for background and basic information about approximation properties and, in particular, the BAP and the uniform approximation property (UAP in short).

Theorem 4.5 ([27], Theorem 2.1). *Let J be an M-ideal in the Banach space E and $\pi : E \to E/J$ the natural quotient map. Let Y be a separable Banach space and $t : Y \to E/J$ be an operator. Assume further that one of the following conditions is satisfied:*

(1) *Y has the λ-AP.*
(2) *J is a Lindenstrauss space.*

Then t can be lifted to E, that is, there is an operator $T : Y \to E$ such that $\pi T = t$. Moreover one can get $\|T\| \leq \lambda\|t\|$ under the assumption (1) and $\|T\| = \|t\|$ under (2).

One has.

Proposition 4.6. *Let J be an M-ideal in a Banach space E.*

(a) *If E is λ-(universally) separably injective, then E/J is λ^2-(universally) separably injective.*
(b) *If E is λ-separably injective, then J is $2\lambda^2$-separably injective.*

When J is a Lindenstrauss space (which is always the case if E is), then the exponent 2 disappears. In particular, if K_1 is a closed subset of the compact space K and $K_0 = K \setminus K_1$ one has:

(c) *If $C(K)$ is λ-(universally) separably injective, then so is $C(K_1)$.*
(d) *If $C(K)$ is λ-separably injective, then $C_0(K_0)$ is 2λ-separably injective.*

Proof. (a) By (the proof of) Proposition 3.4, E^{**} is λ-injective and so it has the λ-AP. Since $E^{**} = J^{**} \oplus_\infty (E/J)^{**}$ we see that also J^{**} and $(E/J)^{**}$ have the λ-AP. Hence both J and (E/J) have the λ-AP. Let Y be a separable subspace of X and $t : Y \longrightarrow E/J$ an operator. Let S be a separable subspace of E/J containing the image of t. By [17, Theorem 9.7] we may assume S has the λ-AP. Let $s : S \longrightarrow E$ be the lifting provided by Theorem 4.5, so that $\|s\| \leq \lambda$. Now, if $T : X \longrightarrow E$ is an extension of st, then $\pi T : X \longrightarrow E/J$ is an extension of t, and this can be achieved with $\|\pi T\| = \|T\| \leq \lambda^2 \|t\|$.

(d) –and (b)–. Let us remark that if S is a subspace of $C(K)$ containing $C_0(K_0)$ and $S/C_0(K_0)$ is separable, then there is a projection $p : S \longrightarrow C_0(K_0)$ of norm at most 2. Indeed, $S/C_0(K_0)$ is a separable subspace of $C(K_1)$ and there is a lifting $s : S/C_0(K_0) \longrightarrow C(K)$, with $\|s\| = 1$, and $p = \mathbf{1}_S - sr$ is the required projection. Now, let $t : Y \longrightarrow C_0(K_0)$ be an operator, where Y is a subspace of a separable Banach space X. Considering t as taking values in $C(K)$, there is an extension $T : X \longrightarrow C(K)$ with $\|T\| \leq \lambda\|t\|$. Let S denote the least closed subspace of $C(K)$ containing the range of T and $C_0(K_0)$ and $p : S \longrightarrow C_0(K_0)$ a projection with $\|p\| \leq 2$. The composition $pT : X \longrightarrow C_0(K_0)$ is an extension of t and clearly, $\|pT\| \leq 2\lambda\|t\|$. $\square$

4.6. Ultraproducts of type $\mathcal{L}_\infty$. Let us briefly recall the definition and some basic properties of ultraproducts of Banach spaces. For a detailed study of this construction at the elementary level needed here we refer the reader to Heinrich's survey paper [28] or Sims' notes [49]. Let I be a set, $\mathcal{U}$ be an ultrafilter on I, and $(X_i)_{i \in I}$ a family of Banach spaces. Then $\ell_\infty(X_i)$ endowed with the supremum norm, is a Banach space, and $c_0^{\mathcal{U}}(X_i) = \{(x_i) \in \ell_\infty(X_i) : \lim_{\mathcal{U}(i)} \|x_i\| = 0\}$ is a closed subspace of $\ell_\infty(X_i)$. The ultraproduct of the spaces $(X_i)_{i \in I}$ following $\mathcal{U}$ is defined as the quotient

$$[X_i]_{\mathcal{U}} = \ell_\infty(X_i)/c_0^{\mathcal{U}}(X_i).$$

We denote by $[(x_i)]$ the element of $[X_i]_\mathcal{U}$ which has the family (x_i) as a representative. It is not difficult to show that $\|[(x_i)]\| = \lim_{\mathcal{U}(i)} \|x_i\|$. In the case $X_i = X$ for all i, we denote the ultraproduct by $X_\mathcal{U}$, and call it the ultrapower of X following $\mathcal{U}$. If $T_i : X_i \to Y_i$ is a uniformly bounded family of operators, the ultraproduct operator $[T_i]_\mathcal{U} : [X_i]_\mathcal{U} \to [Y_i]_\mathcal{U}$ is given by $[T_i]_\mathcal{U}[(x_i)] = [T_i(x_i)]$. Quite clearly, $\|[T_i]_\mathcal{U}\| = \lim_{\mathcal{U}(i)} \|T_i\|$.

Definition 4.7. An ultrafilter $\mathcal{U}$ on a set I is countably incomplete if there is a decreasing sequence (I_n) of subsets of I such that $I_n \in \mathcal{U}$ for all n, and $\bigcap_{n=1}^\infty I_n = \varnothing$.

Notice that $\mathcal{U}$ is countably incomplete if and only if there is a function $n : I \to \mathbb{N}$ such that $n(i) \to \infty$ along $\mathcal{U}$ (equivalently, there is a family $\varepsilon(i)$ of strictly positive numbers converging to zero along $\mathcal{U}$). It is obvious that any countably incomplete ultrafilter is non-principal and also that every non-principal (or free) ultrafilter on $\mathbb{N}$ is countably incomplete. Assuming all free ultrafilters countably incomplete is consistent with ZFC, since the cardinal of a set supporting a free countably complete ultrafilter should be measurable, hence strongly inaccessible.

It is clear that the classes of $\mathcal{L}_{p,\lambda^+}$ spaces are stable under ultraproducts [10, Proposition 1.22]. In the opposite direction, a Banach space is a $\mathcal{L}_{p,\lambda^+}$ space if and only if some (or every) ultrapower is. In particular, a Banach space is an $\mathcal{L}_\infty$ space or a Lindenstrauss space if and only if so are its ultrapowers; see, e.g., [29]. It is possible however to produce a Lindenstrauss space out of non-even-$\mathcal{L}_\infty$-spaces: indeed, if $p(i) \to \infty$ along $\mathcal{U}$, then the ultraproduct $[L_{p(i)}]_\mathcal{U}$ is a Lindenstrauss space (and, in fact, an abstract $\mathcal{M}$-space; see [?, Lemma 3.2]).

The following result about the structure of separable subspaces of ultraproducts of type $\mathcal{L}_\infty$ will be fundamental for us.

Lemma 4.8. *Supppose $[X_i]_\mathcal{U}$ is an $\mathcal{L}_{\infty,\lambda^+}$-space. Then each separable subspace of $[X_i]_\mathcal{U}$ is contained in a subspace of the form $[F_i]_\mathcal{U}$, where $F_i \subset X_i$ is finite dimensional and $\lim_{\mathcal{U}(i)} d(F_i, \ell_\infty^{k(i)}) \leq \lambda$, where $k(i) = \dim F_i$.*

Proof. Let us assume S is an infinite-dimensional separable subspace of $[X_i]_\mathcal{U}$. Let (s^n) be a linearly independent sequence spanning a dense subspace in S and, for each n, let (s_i^n) be a fixed representative of s^n in $\ell_\infty(X_i)$. Let $S^n = \mathrm{span}\{s^1, \ldots, s^n\}$. Since $[X_i]_\mathcal{U}$ is an $\mathcal{L}_{\infty,\lambda^+}$-space there is, for each n, a finite dimensional $F^n \subset [X_i]_\mathcal{U}$ containing S^n with $d(F^n, \ell_\infty^{\dim F^n}) \leq \lambda + 1/n$.

For fixed n, let (f^m) be a basis for F^n containing $s^1, \ldots, s^n$. Choose representatives (f_i^m) such that $f_i^m = s_i^\ell$ if $f^m = s^\ell$. Moreover, let F_i^n be the subspace of X_i spanned by f_i^m for $1 \leq m \leq \dim F^n$.

Let (I_n) be a decreasing sequence of subsets $I_n \in \mathcal{U}$ such that $\bigcap_{n=1}^\infty I_n = \varnothing$. For each integer n put

$$J_n' = \{i \in I : d(F_i^n, \ell_\infty^{\dim F^n}) \leq \lambda + 2/n\} \cap I_n$$

and $J_m = \bigcap_{n \leq m} J_n'$. All these sets are in $\mathcal{U}$. Finally, set $J_\infty = \bigcap_n J_n$. Next we define a function $k : I \to \mathbb{N}$. Set

$$k(i) = \begin{cases} 1 & i \in J_\infty \\ \sup\{n : i \in J_n\} & i \notin J_\infty \end{cases}$$

For each $i \in I$, take $F_i = F_i^{k(i)}$. This is a finite-dimensional subspace of X_i whose Banach-Mazur distance to the corresponding ℓ_∞^k is at most $\lambda + 2/k(i)$. It is clear that $[F_i]_\mathcal{U}$ contains S and also that $k(i) \to \infty$ along $\mathcal{U}$, which completes the proof. $\square$

Theorem 4.9. *Let $(X_i)_{i \in I}$ be a family of Banach spaces such that $[X_i]_\mathcal{U}$ is a $\mathcal{L}_{\infty,\lambda^+}$-space. Then $[X_i]_\mathcal{U}$ is λ-universally separably injective.*

Proof. It is clear that a Banach space is λ-universally separably injective if and only if every separable subspace is contained in some larger λ-universally separably injective subspace. By the previous lemma, everything that has to be proved is:

Lemma 4.10. *For every function $k : I \to \mathbb{N}$, the space $[\ell_\infty^{k(i)}]_\mathcal{U}$ is 1-universally separably injective.*

Proof. Let Γ be the disjoint union of the sets $\{1, 2, \dots, k(i)\}$ viewed as a discrete set. Now observe that $c_0^\mathcal{U}(\ell_\infty^{k(i)})$ is an ideal in $\ell_\infty(\ell_\infty^{k(i)}) = \ell_\infty(\Gamma) = C(\beta\Gamma)$ and apply Proposition 4.6(a). $\qquad\square$

$\hfill\square$

Corollary 4.11. *Let $(X_i)_{i \in I}$ be a family of Banach spaces. If $[X_i]_\mathcal{U}$ is a Lindenstrauss space, then it is 1-universally separably injective.*

Remarks 4.12. Ultraproducts of type $\mathcal{L}_\infty$ are universally separably injective, while an infinite dimensional ultraproduct via a countably incomplete ultrafilter is never injective (see [31, Theorem 2.6]; and also [49, Section 8].

5. TWO CHARACTERIZATIONS OF UNIVERSALLY SEPARABLY INJECTIVE SPACES

In Proposition 3.4 (b) it was proved that universally separably injective spaces contain ℓ_∞. Much more is indeed true:

Theorem 5.1. *An infinite-dimensional Banach space is universally separably injective if and only if it is ℓ_∞-upper-saturated.*

Proof. The sufficiency is a consequence of the injectivity of ℓ_∞. In order to show the necessity, let Y be a separable subspace of a universally separably injective space X. We consider a subspace Y_0 of ℓ_∞ isomorphic to Y and an isomorphism $t : Y_0 \to Y$. We can find projections p on X and q on ℓ_∞ such that $Y \subset \ker p, Y_0 \subset \ker q$, and both p and q have range isomorphic to ℓ_∞.

Indeed, let $\pi : X \to X/Y$ be the quotient map. Since X contains ℓ_∞ and Y is separable, π is not weakly compact so, by Proposition 3.4(b), there exists a subspace M of X isomorphic to ℓ_∞ where π is an isomorphism. Now $X/Y = \pi(M) \oplus N$, with N a closed subspace. Hence $X = M \oplus \pi^{-1}(N)$, and it is enough to take p as the projection with range M and kernel $\pi^{-1}(N)$.

Since $\ker p$ and $\ker q$ are universally separably injective spaces, we can take operators $u : X \to \ker q$ and $v : \ell_\infty \to \ker p$ such that $v = t$ on Y_0 and $u = t^{-1}$ on Y.

Let $w : \ell_\infty \to \operatorname{ran} p$ be an operator satisfying $\|w(x)\| \geq \|x\|$ for all $x \in \ell_\infty$. We will show that the operator
$$T = v + w(\mathbf{1}_{\ell_\infty} - uv) : \ell_\infty \longrightarrow X$$
is an isomorphism (into). This suffices to end the proof since $\operatorname{ran} T$ is isomorphic to ℓ_∞ and both T and v agree with t on Y_0, so $Y \subset \operatorname{ran} T \subset X$.

Since $\operatorname{ran} v \subset \ker p$ and $\operatorname{ran} w \subset \operatorname{ran} p$, there exists $C > 0$ such that
$$\|Tx\| \geq C \max\{\|v(x)\|, \|w(\mathbf{1}_{\ell_\infty} - uv)x\|\} \qquad (x \in \ell_\infty).$$
Now, if $\|vx\| < (2\|u\|)^{-1}\|x\|$, then $\|uvx\| < \frac{1}{2}\|x\|$; hence
$$\|w(\mathbf{1}_{\ell_\infty} - uv)x\| \geq \|(\mathbf{1}_{\ell_\infty} - uv)x\| > \frac{1}{2}\|x\|.$$
Thus $\|Tx\| \geq C(2\|u\|)^{-1}\|x\|$ for every $x \in X$. $\hfill\square$

We can now complete the proof of Proposition 3.6(3) and show that the class of universally separably injective spaces has the 3-space property.

Proposition 5.2. *The class of universally separably injective spaces has the 3-space property.*

Proof. By Theorem 5.1 one has to show that being ℓ_∞-upper-saturated is a 3-space property.

Let $0 \longrightarrow Y \longrightarrow X \stackrel{q}{\longrightarrow} Z \longrightarrow 0$ be an exact sequence in which both Y, Z are ℓ_∞-uppersaturated, and let S be a separable subspace of X. It is not hard to find separable subspaces S_0, S_1 of X such that $S \subset S_1$ and $S_1/S_0 = [q(S)]$. Let Y_∞ be a copy of ℓ_∞ inside Y containing S_0. By the injectivity of ℓ_∞, S is contained in the subspace $Y_\infty \oplus [q(S)]$ of X. And since there exists a copy Z_∞ of ℓ_∞ containing $[q(S)]$, S is therefore contained in the subspace $Y_\infty \oplus Z_\infty$ of X, which is isomorphic to ℓ_∞. $\qquad\square$

A homological characterization of universally separably injective spaces is also possible. We need first to show:

Proposition 5.3. *If U is a universally separably injective space then* $\mathrm{Ext}(\ell_\infty, U) = 0$.

Proof. James's well known distortion theorem for ℓ_1 (resp. c_0) asserts that a Banach space containing a copy of ℓ_1 (resp. c_0) also contains an almost isometric copy of ℓ_1 (resp. c_0). Not so well known is Partington's distortion theorem for ℓ_∞ [45]: a Banach space containing ℓ_∞ contains an almost isometric copy of ℓ_∞ (see also Dowling [26]). This last copy will therefore be, say, 2-complemented.

Let Γ denote the set of all the 2-isomorphic copies of ℓ_∞ inside ℓ_∞. For each $E \in \Gamma$ let $\iota_E : E \to \ell_\infty$ be the canonical embedding, p_E a projection onto E of norm at most 2 and $u_E : E \to \ell_\infty$ an isomorphism. Assume that a nontrivial exact sequence

$$0 \longrightarrow U \longrightarrow X \longrightarrow \ell_\infty \longrightarrow 0$$

exists. We consider, for each $E \in \Gamma$, a copy of the preceding sequence, and form the product of all these copies $0 \longrightarrow \ell_\infty(\Gamma, U) \longrightarrow \ell_\infty(\Gamma, X) \longrightarrow \ell_\infty(\Gamma, \ell_\infty) \longrightarrow 0$. Let us consider the embedding $J : \ell_\infty \to \ell_\infty(\Gamma, \ell_\infty)$ defined as $J(x)(E) = u_E p_E(x)$ and then form the pull-back sequence

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \ell_\infty(\Gamma, U) & \longrightarrow & \ell_\infty(\Gamma, X) & \longrightarrow & \ell_\infty(\Gamma, \ell_\infty) & \longrightarrow & 0 \\
& & \| & & \uparrow & & \uparrow{\scriptstyle J} & & \\
0 & \longrightarrow & \ell_\infty(\Gamma, U) & \longrightarrow & \mathrm{PB} & \stackrel{q}{\longrightarrow} & \ell_\infty & \longrightarrow & 0
\end{array}
$$

Let us show that q cannot be an isomorphism on a copy of ℓ_∞. Otherwise, it would be an isomorphism on some $E \in \Gamma$ and thus the new pull-back sequence

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \ell_\infty(\Gamma, U) & \longrightarrow & \mathrm{PB} & \stackrel{q}{\longrightarrow} & \ell_\infty & \longrightarrow & 0 \\
& & \| & & \uparrow & & \uparrow{\scriptstyle \iota_E} & & \\
0 & \longrightarrow & \ell_\infty(\Gamma, U) & \longrightarrow & \mathrm{PB}_E & \longrightarrow & E & \longrightarrow & 0
\end{array}
$$

would split. And therefore the same would be true making push-out with the canonical projection $\pi_E : \ell_\infty(\Gamma, U) \to U$ onto the E-th copy of U:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \ell_\infty(\Gamma, U) & \longrightarrow & \mathrm{PB}_E & \longrightarrow & E & \longrightarrow & 0 \\
& & {\scriptstyle \pi_E}\downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & U & \longrightarrow & \mathrm{PO}_E & \longrightarrow & E & \longrightarrow & 0
\end{array}
$$

But it is not hard to see that new pull-back with u_E^{-1}

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & U & \longrightarrow & \mathrm{PO}_E & \longrightarrow & E & \longrightarrow & 0 \\
& & \| & & \uparrow & & \uparrow{\scriptstyle u_E^{-1}} & & \\
0 & \longrightarrow & U & \longrightarrow & X & \longrightarrow & \ell_\infty & \longrightarrow & 0
\end{array}
$$

produces exactly the starting sequence which, by assumption, was nontrivial.

However, the space PB should be universally separably injective by Proposition 3.6(3), hence it must have Rosenthal's property (V), by Proposition 3.4(b). This contradiction shows that the starting nontrivial sequence cannot exist. □

We are thus ready to prove:

Theorem 5.4. *A Banach space U is universally separably injective if and only if for every separable space S one has $\mathrm{Ext}(\ell_\infty/S, U) = 0$.*

Proof. Let S be separable and let U be universally separably injective. Applying $\mathfrak{L}(-, U)$ to the sequence $0 \to S \to \ell_\infty \to \ell_\infty/S \to 0$ one gets the exact sequence

$$\ldots \longrightarrow \mathfrak{L}(\ell_\infty, U) \longrightarrow \mathfrak{L}(S, U) \longrightarrow \mathrm{Ext}(\ell_\infty/S, U) \longrightarrow \mathrm{Ext}(\ell_\infty, U)$$

Since $\mathrm{Ext}(\ell_\infty, U) = 0$, one obtains that every exact sequence $0 \to U \to X \to \ell_\infty/S \to 0$ fits in a push-out diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & S & \longrightarrow & \ell_\infty & \longrightarrow & \ell_\infty/S & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & U & \longrightarrow & X & \longrightarrow & \ell_\infty/S & \longrightarrow & 0.
\end{array}
$$

Since U is universally separably injective, the lower sequence splits.

The converse is clear: every operator $t : S \to U$ from a separable Banach space into a space U produces a push-out diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & S & \longrightarrow & \ell_\infty & \longrightarrow & \ell_\infty/S & \longrightarrow & 0 \\
 & & {\scriptstyle t}\downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & U & \longrightarrow & \mathrm{PO} & \longrightarrow & \ell_\infty/S & \longrightarrow & 0.
\end{array}
$$

The lower sequence splits by the assumption $\mathrm{Ext}(\ell_\infty/S, U) = 0$ and so t extends to ℓ_∞, according to the splitting criterion for push-out sequences. □

Which leads to the unexpected:

Corollary 5.5. $\mathrm{Ext}(\ell_\infty/c_0, \ell_\infty/c_0) = 0$; *i.e., every short exact sequence $0 \to \ell_\infty/c_0 \to X \to \ell_\infty/c_0 \to 0$ splits.*

This result provides a new solution for equation $\mathrm{Ext}(X, X) = 0$. The other three previously known types of solutions are: c_0 (by Sobczyk theorem), the injective spaces (by the very definition) and the $L_1(\mu)$-spaces (by Lindenstrauss' lifting).

Also, in rough contrast with Proposition 5.2, one has:

Corollary 5.6. *Rosenthal's property (V) is not a 3-space property*

Proof. With the same construction as above, start with a nontrivial exact sequence $0 \to \ell_2 \to E \to \ell_\infty \to 0$ (see [13, Section 4.2]) and construct an exact sequence

$$0 \longrightarrow \ell_\infty(\Gamma, \ell_2) \longrightarrow X \xrightarrow{q} \ell_\infty \longrightarrow 0,$$

where q cannot be an isomorphism on a copy of ℓ_∞. So X has not Rosenthal's property (V). The space $\ell_\infty(\Gamma, \ell_2)$ has Rosenthal's property (V) as a quotient of $\ell_\infty(\Gamma, \ell_\infty) = \ell_\infty(\mathbb{N} \times \Gamma)$, since the property obviously passes to quotients. □

It is not however true that $\mathrm{Ext}(U, V) = 0$ for all universally separably injective spaces U and V as any exact sequence $0 \to U \to \ell_\infty(\Gamma) \to \ell_\infty(\Gamma)/U \to 0$ in which U is a universally separably injective non-injective space shows.

6. 1-SEPARABLY INJECTIVE SPACES

Our first result here establishes a major difference between 1-separably injective and general separably injective spaces: 1-separably injective spaces must be Grothendieck (hence they cannot be separable or WCG) while a 2-separably injective space, such as c_0, can be even separable. The following lemma due to Lindenstrauss [40, p. 221, proof of (i) $\Rightarrow$ (v)] provides a quite useful technique.

Lemma 6.1. *Let E be a 1-separably injective space and Y a separable subspace of X, with* $\operatorname{dens} X = \aleph_1$. *Then every operator $t : Y \to E$ can be extended to an operator $T : X \to E$ with the same norm.*

This yields

Proposition 6.2. *Under* CH *every 1-separably injective Banach space is universally 1-separably injective and therefore a Grothendieck space.*

Proof. Let E be 1-separably injective, X an arbitrary Banach space and $t : Y \to E$ an operator, where Y is a separable subspace of X. Let $[t(Y)]$ be the closure of the image of t. This is a separable subspace of E and so there is an isometric embedding $u : [t(Y)] \to \ell_\infty$. As ℓ_∞ is 1-injective there is an operator $T : X \to \ell_\infty$ whose restriction to Y agrees with ut. Thus it suffices to extend the inclusion of $[t(Y)]$ into E to ℓ_∞. But, under CH, the density character of ℓ_∞ is $\aleph_1$ and the preceding Lemma applies. The 'therefore' part is now a consequence of Proposition 3.4(b). □

The "therefore" part survives in ZFC:

Theorem 6.3. *Every 1-separably injective space is a Grothendieck and a Lindenstrauss space.*

Proof. The proof of Proposition 3.4 yields that 1-separably injective spaces are of type $\mathcal{L}_{\infty,1+}$, that is, Lindenstrauss spaces. It remains to prove that a 1-separably injective space E must be Grothendieck. It suffices to show that c_0 is not complemented in E, so let $\jmath : c_0 \longrightarrow E$ be an embedding. Consider an almost-disjoint family $\mathcal{M}$ of size $\aleph_1$ formed by infinite subsets of $\mathbb{N}$ and construct the associated Johnson-Lindenstrauss twisted sum space

$$0 \longrightarrow c_0 \longrightarrow C(\Delta_\mathcal{M}) \longrightarrow c_0(\aleph_1) \longrightarrow 0.$$

Observe that the space $C(\Delta_\mathcal{M})$ has density character $\aleph_1$, we have therefore a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & c_0 & \longrightarrow & C(\Delta_\mathcal{M}) & \longrightarrow & c_0(\aleph_1) & \longrightarrow & 0 \\
& & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & c_0 & \overset{\jmath}{\longrightarrow} & E & \longrightarrow & E/\jmath(c_0) & \longrightarrow & 0.
\end{array}
$$

If c_0 was complemented in E then it would be complemented in $C(\Delta_\mathcal{M})$ as well, which is not. □

Proposition 6.2 leads to the question about the necessity of the hypothesis CH. We will prove now that it cannot be dropped.

6.1. A 1-separably injective but not 1-universally separably injective $C(K)$.

Lemma 6.4. *Let K, L, M be compact spaces and let $f : K \longrightarrow M$ be a continuous map, with $\jmath = f^\circ : C(M) \longrightarrow C(K)$ its induced operator, and let $\imath : C(M) \longrightarrow C(L)$ be a positive norm one operator. Suppose that $S : C(L) \longrightarrow C(K)$ is an operator with $\|S\| = 1$ and $S\imath = \jmath$. Then S is a positive operator.*

Proof. Obviously $S \geq 0$ if and only if $S^*\delta_x \geq 0$ for all $x \in K$, where δ_x is the unit mass at x and $S^* : C(K)^* \to C(L)^*$ is the adjoint operator. Fix $x \in K$. By Riesz theorem we have that $S^*\delta_x = \mu$

is a measure of total variation $\|\mu\| \leq 1$. Let $\mu = \mu^+ - \mu^-$ be the Hahn-Jordan decomposition of μ, so that $\|\mu\| = \|\mu^+\| + \|\mu^-\|$, with $\mu^+, \mu^- \geq 0$. We have that $\delta_{f(x)} = \jmath^* \delta_x = \imath^* S^* \delta_x = \imath^* \mu$, thus

$$\delta_{f(x)} = \imath^* \mu^+ - \imath^* \mu^- \quad \text{and} \quad \|\delta_{f(x)}\| = \|\imath^* \mu^+\| + \|\imath^* \mu^-\|.$$

Since $\imath$ is a positive operator these imply that the above is the Hahn-Jordan decomposition of $\delta_{f(x)}$ and so $\imath^* \mu^- = 0$, hence $\mu^- = 0$. $\qquad\square$

Definition 6.5. Let L be a zero-dimensional compact space. An $\aleph_2$-Lusin family on L is a family $\mathcal{F}$ of pairwise disjoint nonempty clopen subsets of L with $|\mathcal{F}| = \aleph_2$, such that whenever $\mathcal{G}$ and $\mathcal{H}$ are subfamilies of $\mathcal{F}$ with $|\mathcal{G}| = |\mathcal{H}| = \aleph_2$, then

$$\overline{\bigcup \{G \in \mathcal{G}\}} \cap \overline{\bigcup \{G \in \mathcal{H}\}} \neq \varnothing.$$

The following lemma shows the consistency of the existence of an $\aleph_2$-Lusin family on $\mathbb{N}^*$. This is rather folklore of set-theory, but we did not find a reference so we state it and give a hint of the proof.

Lemma 6.6. *Under* MA *and the assumption* $\mathfrak{c} = \aleph_2$ *there exists an* $\aleph_2$-*Lusin family on* $\mathbb{N}^*$.

Proof. By Stone duality, since the Boolean algebra associated to $\mathbb{N}^*$ is $\wp(\mathbb{N})/\mathrm{fin}$, an $\aleph_2$-Lusin family on $\mathbb{N}^*$ is all the same as an almost disjoint family $\{A_\alpha\}_{\alpha<\omega_2}$ of infinite subsets of $\mathbb{N}$ such that for every $B \subset \mathbb{N}$ either $\{\alpha : |A_\alpha \setminus B| \text{ is finite}\}$ or $\{\alpha : |A_\alpha \cap B| \text{ is finite}\}$ has cardinality $< \aleph_2$. Let $\{B_\alpha : \alpha < \omega_2\}$ be an enumeration of all infinite subsets of $\mathbb{N}$. We construct the sets A_α inductively on α. Suppose A_γ has been constructed for $\gamma < \alpha$. We define a forcing notion $\mathbb{P}$ whose conditions are pairs $p = (f_p, F_p)$ where f_p is a $\{0,1\}$-valued function on a finite subset $\mathrm{dom}(f_p)$ of $\mathbb{N}$ and F_p is a finite subset of α. The order relation is that $p < q$ if f_p extends f_q, $F_p \supset F_q$ and f_p vanishes in $A_\gamma \setminus \mathrm{dom}(f_q)$ for $\gamma \in F_q$. One checks that this forcing is ccc. Hence, by MA, using a big enough generic filter the forcing provides an infinite set $A_\alpha \subset \mathbb{N}$ such that, for all $\gamma < \alpha$,

(1) $A_\alpha \cap A_\gamma$ is finite, and
(2) If B_γ is not contained in any finite union of A_δ's, then $A_\alpha \cap B_\gamma$ is infinite.

$\qquad\square$

Theorem 6.7. *It is consistent that there exists a compact space K for which the Banach space $C(K)$ is 1-separably injective but not universally 1-separably injective.*

Proof. We will suppose that $\mathfrak{c} = \aleph_2$ and that there exists an $\aleph_2$-Lusin family in $\mathbb{N}^*$. Under these hypotheses, let K be the Stone dual compact space of the Cohen-Parovičenko Boolean algebra of [25, Theorem 5.6]. The definition of that Boolean algebra implies that K is an F-space and thus $C(K)$ is 1-separably injective by Theorem 4.2. We show that it is not universally 1-separably injective. The argument follows the scheme of [25, Theorem 5.10], where they prove that K does not map onto $\beta\mathbb{N}$, but we use $\aleph_2$-Lusin families instead of ω_2-chains because they fit better in the functional analytic context.

Let $\{U_n : n \in \mathbb{N}\}$ be a sequence of pairwise disjoint clopen subsets of K, and let $U = \bigcup_n U_n$. Let $c \subset \ell_\infty$ be the Banach space of convergent sequences, and $t : c \longrightarrow C(K)$ be the operator given by $t(z)(x)z_n$ if $x \in U_n$ and $t(z)(x) = \lim z_n$ if $x \notin U$.

If $C(K)$ were universally 1-separably injective, we should have an extension $T : \ell_\infty \longrightarrow C(K)$ of t with $\|T\| = 1$. We shall derive a contradiction from the existence of such operator.

Notice that the conditions of Lemma 6.4 are applied, so T is positive (observe that $c = C(\mathbb{N} \cup \{\infty\})$ and $T = f^\circ$ where $f : K \longrightarrow \mathbb{N} \cup \{\infty\}$ is given by $f(x) = n$ if $x \in U_n$ and $f(x) = \infty$ if $x \notin U$).

For every $A \subset \mathbb{N}$ we will denote $[A] = \overline{A}^{\beta\mathbb{N}} \setminus \mathbb{N}$. The clopen subsets of $\mathbb{N}^*$ are exactly the sets of the form $[A]$, and we have that $[A] = [B]$ if and only if $(A \setminus B) \cup (B \setminus A)$ is finite.

Let $\mathcal{F}$ be an $\aleph_2$-Lusin family in $\mathbb{N}^*$. For $F = [A] \in \mathcal{F}$ and $0 < \varepsilon < \frac{1}{2}$, let

$$F_\varepsilon = \{x \in K \setminus U : T(1_A)(x) > 1 - \varepsilon\}.$$

Let us remark that F_ε depends only on F and not on the choice of A. This is because if $[A] = [B]$, then $1_A - 1_B \in c_0$, hence $T(1_A - 1_B) = t(1_A - 1_B)$ which vanishes out of U, so $T(1_A)|_{K \setminus U} = T(1_B)|_{K \setminus U}$.

CLAIM 1. If $\delta < \varepsilon$ and $F \in \mathcal{F}$, then $\overline{F_\delta} \subset F_\varepsilon$.

CLAIM 2. $F_\varepsilon \cap G_\varepsilon = \varnothing$ for every $F \neq G$.

Proof of Claim 2. Since $F \cap G = \emptyset$ we can choose $A, B \subset \mathbb{N}$ such that $F = [A]$, $G = [B]$ and $A \cap B = \emptyset$. If $x \in F_\varepsilon \cap G_\varepsilon$, $\bar{T}(1_A + 1_B)(x) > 2 - 2\varepsilon > 1$ which is a contradiction because $1_A + 1_B = 1_{A \cup B}$ and $\|\bar{T}(1_{A \cup B})\| \leq \|\bar{T}\| \|1_{A \cup B}\| = 1$. *End of the Proof of Claim 2.*

For every $F \in \mathcal{F}$, let $\tilde{F}$ be a clopen subset of $K \setminus U$ such that $\overline{F_{0.2}} \subset \tilde{F} \subset F_{0.3}$. By the preceding claims, this is a disjoint family of clopen sets. It follows from Proposition 2.6 and Corollary 5.12 in [25] that $K \setminus U$ does not contain any $\aleph_2$-Lusin family. Therefore we can find $\mathcal{G}, \mathcal{H} \subset \mathcal{F}$ with $|\mathcal{G}| = |\mathcal{H}| = \aleph_2$ such that

$$\overline{\bigcup\{\tilde{G} : G \in \mathcal{G}\}} \cap \overline{\bigcup\{\tilde{H} : H \in \mathcal{H}\}} = \varnothing.$$

Now, for every $n \in \mathbb{N}$ choose a point $p_n \in U_n$. Let $g : \beta\mathbb{N} \longrightarrow K$ be a continuous function such that $g(n) = p_n$.

CLAIM 3. For $u \in \beta\mathbb{N}$, $A \subset \mathbb{N}$, $T(1_A)(g(u)) = \begin{cases} 1, & \text{if } u \in [A]; \\ 0, & \text{if } u \notin [A]. \end{cases}$

Proof of Claim 3. It is enough to check it for $u = n \in \mathbb{N}$. This is a consequence of the fact that T is positive, because if $m \in A$, $n \notin A$, then $0 \leq t(1_m) \leq T(1_A) \leq t(1_{\mathbb{N} \setminus \{n\}}) \leq 1$. *End of the Proof of Claim 3.*

The function g is one-to-one because

$$\overline{\{p_n : n \in A\}} \cap \overline{\{p_n : n \notin A\}} = \varnothing$$

for every $A \subset \mathbb{N}$, as the function $T(1_A)$ separates these sets. On the other hand, as a consequence of Claim 3 above, for every $F \in \mathcal{F}$ and every ε, $g^{-1}(F_\varepsilon) = F$, and also $g^{-1}(\tilde{F}) = F$. These facts make the families $\mathcal{H}$ and $\mathcal{G}$ above to contradict that $\mathcal{F}$ is an $\aleph_2$-Lusin family in $\mathbb{N}^*$. $\square$

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